Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Iterative Techniques for the Solution of Large Linear Systems in Computational Aerodynamics

A. Bratkovich* and F. J. Marshall†
Purdue University, West Lafayette, Ind.

Nomenclature

A_{ij} = aerodynamic influence coefficient matrix
B = blockage factor
$C_i^{(q)}$ = $S_i^{(q)}/\alpha_i^{(q)}\alpha_i^{(q-1)}$
D_{ij} = diagonal block influence coefficient matrix
E_{ij} = $L_{ij}+U_{ij}$
L_{ij} = lower off-diagonal block influence coefficient matrix
N = number of singularities
Q = number of iteration steps for convergence
$S_i^{(q)}$ = $(\gamma_i^{(q)}-\gamma_i^{(q-1)})(\gamma_i^{(q-1)}-\gamma_i^{(q-2)})$
$SG^{(q)}$ = $\sum_{i=1}^{N} |\gamma_i^{(q)}-\gamma_i^{(q-1)}|$
U_{ij} = upper off-diagonal block influence coefficient matrix
$\alpha,\alpha_1,\alpha_2,\alpha(i)^{(q)}$ = relaxation parameters
γ_i = singularity distribution strengths
$\bar{\gamma}_i$ = Gauss-Seidel approximation
ω_j = normal velocities
$(\ )^{(q)}$ = qth iteration step
J = Jacobi
GS = Gauss-Seidel
SOR = successive over-relaxation
$CSOR$ = controlled successive over-relaxation
$(\ )_b$ = blocked

Introduction

THE method of aerodynamic influence coefficients[1] has developed into an important tool for the analysis and design of aircraft. These collocation techniques give rise to large systems of linear equations whose solution often requires a major part of the computational time for the method which may impose constraints on the complexity of the configuration. Consequently, it would be desirable to have an efficient, generalized solution technique for this class of problems.

Iterative approaches have proven must successful in solving large linear systems. There are a variety of such techniques[2-6] available, but most of them require the coefficient matrix to have special properties. The aerodynamic influence coefficient matrix, A_{ij}, is not symmetric, in general has no zero elements, and does not possess even weak diagonal dominance. However, the magnitudes of A_{ij} are generally greater in the neighborhood of the diagonal allowing for blocked techniques. The problem is to find an efficient solution method for such a system that is applicable to a wide class of configurations.

Received August 19, 1974. This work was partially supported by NASA Grant NGR 15-005-188. The authors wish to thank H. Morgan and C. Fox, NASA-Langley, for their substantial contributions.
Index categories: Computer Technology and Computer Simulation Techniques.

*Graduate Student, School of Aeronautics and astronautics.
†Professor, School of Aeronautics and Astronautics. Member AIAA.

Method

The system of equations to be solved (as in Ref. 1) is

$$A_{ij}\gamma_i=\omega_j, i,j=1,2,\ldots N \tag{1}$$

Here the aerodynamic singularity distribution strengths, γ_i for the N panels of a given configuration are related to the normal velocities, ω_j, at the N control points by a matrix of influence coefficients, A_{ij}. A scheme must be developed to find the γ_i.

The proposed iterative technique is given in the context of known techniques. Letting

$$D_{ij}=A_{ij}\quad i-B\leq j\leq i+B;\ U_{ij}=A_{ij}\quad i+B+1\leq j;$$
$$L_{ij}=A_{ij}\quad j\leq i-B-1;\ E_{ij}=U_{ij}+L_{ij} \tag{2}$$

such that $A_{ij}=D_{ij}+E_{ij}=D_{ij}+U_{ij}+L_{ij}$, Eq. (1) may be written as

$$\gamma_i=D_{ij}^{-1}[\omega_j-U_{jk}\gamma_k-L_{jk}\gamma_k] \tag{3}$$

The iteration techniques of block Jacobi (J_b), block Gauss-Seidel (GS_b) and block Successive Over-Relaxation (SOR_b) all start with the same first step ($q=1$),

$$\gamma_i^{(1)}=D_{ij}^{-1}[\omega_j] \tag{4}$$

and for $q\geq 2$, each technique is given by

$$J_b{:}\gamma_i^{(q)}=D_{ij}^{-1}[\omega_j-E_{jk}\gamma_k^{(q-1)}]$$
$$GS_b{:}\gamma_i^{(q)}=D_{ij}^{-1}[\omega_j-U_{jk}\gamma_k^{(q)}-L_{jk}\gamma_k^{(q-1)}] \tag{5}$$
$$SOR_b{:}\gamma_i^{(q)}=\alpha D_{ij}^{-1}(\omega_{ij}-U_{jk}\gamma_k^{(q)}-L_{jk}\gamma_k^{(q-1)}]$$
$$+(1-\alpha)\gamma_i^{(q-1)}$$

Now for SOR_b, the relaxation factor, α, is not given but has to be determined for each particular case. A priori estimation of the optimal relaxation factor for SOR_b is difficult since there is limited information readily available about A_{ij} and its derivative matrices. Thus, SOR_b has to be executed a number of times before an acceptable value of the relaxation factor could be found resulting in a loss in overall computational efficiency. In particular, arbitrary values of α with SOR_b could lead to divergence. (In the following, implied computational times for the SOR_b technique do not include the computational times incurred in the trial-and-error runs performed to obtain the near-optimal value of $\alpha=1.1$ for the particular case to be discussed).

The proposed technique, termed block "controlled successive over-relaxation" ($CSOR_b$), is given by

$$\gamma_i{}^{(1)} = D_{ij}{}^{-1}[\omega_j]$$

$$\gamma_i{}^{(q)} = \alpha(i)^{(q)}\bar{\gamma}_i{}^{(q)} + (1-\alpha(i)^{(q)})\gamma_i{}^{(q-1)}, q \geq 2$$

where

$$\bar{\gamma}_i{}^{(q)} = D_{ij}{}^{-1}[\omega_j - U_{ij}\gamma_j{}^{(q)} - L_{ij}\gamma_j{}^{(q-1)}] \tag{6}$$

$$\alpha(i)^{(q)} = \alpha_1 > 1 \text{ if } C_i{}^{(q)} \geq 0$$

$$= \alpha_2 < 1 \text{ if } C_i{}^{(q)} < 0$$

$$C_i{}^{(q)} = (\bar{\gamma}_i{}^{(q)} - \gamma_i{}^{(q-1)})(\bar{\gamma}_i{}^{(q-1)} - \gamma_i{}^{(q-2)})$$

and

$$\alpha_1 = 1.1;\ \alpha_2 = 0.9$$

[The blockage factor, B, is given in Ref. 1]. $CSOR_b$ is obviously a derivative of SOR_b. They differ in that the relaxation factor, α, in SOR_b must first be determined by trial-and-error runs for the particular configuration while in $CSOR_b$, an algorithm is given for $\alpha = \alpha(i)^{(q)}$, now a function of the singularity and the iteration step.

The rationale for this algorithm, a feedback control mechanism to accelerate convergence, is as follows: The behavior of the solution over the iteration process is given by the sign of the function

$$S_i{}^{(q)} = (\gamma_i{}^{(q)} - \gamma_i{}^{(q-1)})(\gamma_i{}^{(q-1)} - \gamma_i{}^{(q-2)}) \tag{7}$$

such that $S_i{}^{(q)} > 0$ implies a monotonic behavior (see Fig. 1) and $S_i{}^{(q)} < 0$ implies an oscillatory behavior (see Fig. 2). ($S_i{}^{(q)} = 0$ is taken to be monotonic.) Thus, if the solution is behaving in a monotonic fashion, $S_i{}^{(q)} > 0$, then the solution procedure may be accelerated by choosing an $\alpha > 1$. If the solution is behaving in an oscillatory fashion then an $\alpha < 1$ would accelerate the procedure.

However, the computation of $C_i{}^{(q)}$ may be performed more conveniently[1] than $S_i{}^{(q)}$. Since

$$C_i{}^{(q)} = \frac{S_i{}^{(q)}}{\alpha_i{}^{(q)}\alpha_i{}^{(q-1)}} \tag{8}$$

then $C_i{}^{(q)} \gtreqless 0$ implies monotonic or oscillatory behavior and $C_i{}^{(q)}$ is used to indicate solution behavior.

Results

The performance of the block controlled successive over-relaxation method ($CSOR_b$) was compared to several other iterative techniques, namely block Jacobi (J_b), block Gauss-Siedel (GS_b) and block successive over-relaxation (SOR_b) by testing a number of linear systems corresponding to a variety of different geometrical configurations, Mach numbers and angles of attack. The linear systems are those given by the method of Ref. 1.

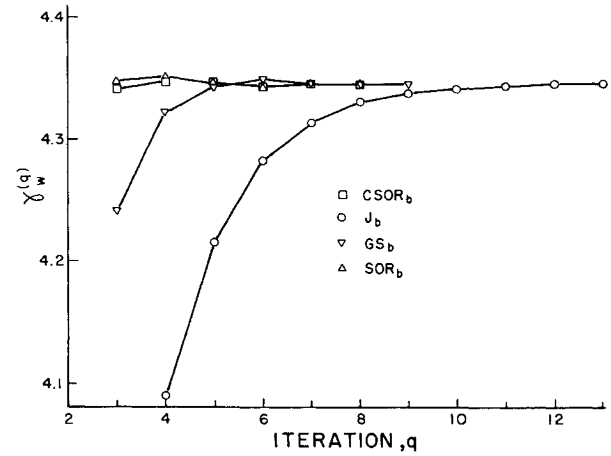

Fig. 1 Monotonic solution behavior with iteration step.

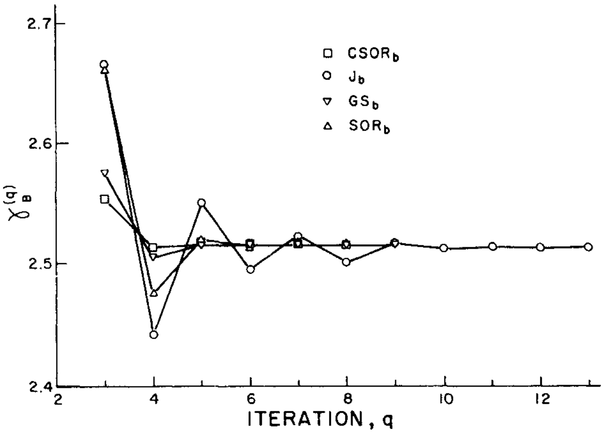

Fig. 2 Oscillatory solution behavior with iteration step.

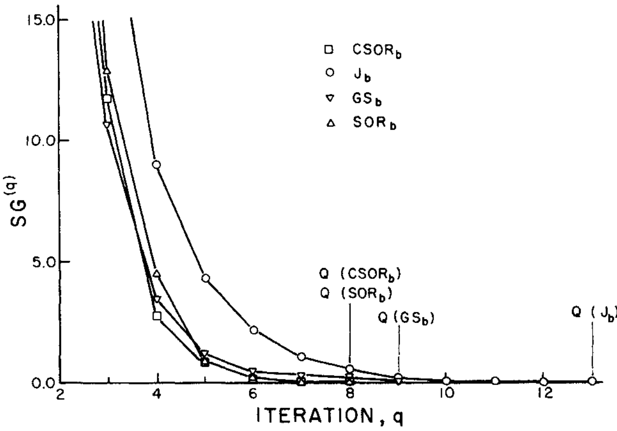

Fig. 3 Total solution behavior with iteration step.

The accelerated convergence of the $CSOR_b$ method is illustrated in the case of a V/STOL aircraft[7] which was run with 342 singularity panels at Mach number =0.2 and an angle of attack of 5°.

Figures 1 and 2 demonstrate the monotonic and oscillatory modes of convergence, respectively, for particular γ_i. [Since all techniques yield the same $\gamma_i{}^{(1)}$, only $\gamma_i{}^{(q)}, q \geq 3$ are presented for clarity.] It is seen that $CSOR_b$ is superior to the remaining three in both modes. Note also that in the context of $CSOR_b$ and SOR_b being generalizations of GS_b, $CSOR_b$ is an improvement over GS_b in both modes while SOR_b is an improvement over GS_b only in the monotonic mode (Fig. 1).

Convergence for the iteration sequence is that of Ref. 1.

$$|\gamma_i{}^{(Q-1)} - \gamma_i{}^{(Q-2)}| > 0.001$$

$$|\gamma_i{}^{(Q)} - \gamma_i{}^{(Q-1)}| < 0.001 \tag{9}$$

for all i. Thus, an indication of the total behavior of the iteration sequence is given by

$$SG^{(q)} = \sum_{i=1}^{N} |\gamma_i{}^{(q)} - \gamma_i{}^{(q-1)}| \tag{10}$$

Figure 3 shows this behavior for all four techniques. The acceleration induced by $CSOR_b$ is demonstrated.

Since Q is the total number of iteration steps computed, QN^2 is an indication of computational times. For the four techniques $QN^2 \times 10^{-5} = 9.3$, 10.5, 20.9, for $CSOR_b$, GS_b, and J_b, respectively and $QN^2.10^{-5} > 9.3$ for SOR_b due to computational times incurred in determining the optimum value of $\alpha(=1.1$ in this case).

Testing of the four techniques for other configurations and Mach numbers, subsonic and supersonic, have been performed within the methods of Ref. 1 with essentially equivalent results. Thus, it has been demonstrated that the use of $CSOR_b$ results in material savings in computational times over the use of J_b, GS_b, and SOR_b.

References

[1] Woodward, F. A., "An Improved Method for the Aerodynamic Analysis of Wing-Body-Tail Configurations in Subsonic and Supersonic Flow," CR-2228, May 1973, NASA.

[2] Young, D. M., "Iterative Solution of Large Linear Systems," Academic Press, New York, 1971.

[3] Varga, R. S., "Matrix Iterative Analysis," Prentice-Hall, Englewood, Cliffs, N.J., 1962.

[4] Carre, B. A., "The Determination of the Optimum Accelerating Factor for Successive Over-Relaxation," *The Computer Journal,* Vol. 4, April 1961, pp. 73-78.

[5] Hageman, L. A. and Kellog, R. B., "Estimating Optimum Over-Relaxation Parameters," *Mathematics of Computation,* Vol. 22, Jan. 1968, pp. 60-68.

[6] Rigler, A. K., "Estimation of the Successive Over-Relaxation Factor," *Mathematics of Computation,* Vol. 19, April 1965, pp. 302-307.

[7] Thomas, J. L., Hoad, D. L., and Croom, D. R., "Aerodynamic Effects of Five Lift-Fan Pod Arrangements on an Unpowered V/STOL Transport," TN D-7199, June 1973, NASA.

Prediction of Airfoil Tone Frequencies

Martin R. Fink*

United Aircraft Research Laboratories, East Hartford, Conn.

AIRFOILS tested at low Reynolds numbers and moderate incidence are found[1-5] to radiate sound as a discrete tone. In one of those studies,[5] it was claimed that this tone is associated with the development of a Karman vortex street in the turbulent near wake. Thus the tone Strouhal number was taken as 0.281 referenced to vortex street width. Measured 1/3 octave frequency bands for the tone were matched by arbitrarily assuming a vortex street width of 0.6 times the sum of calculated upper-surface and lower-surface turbulent boundary-layer thickness at the trailing edge. If this is the correct physical mechanism, airfoil tone noise should occur not just in model tests but in flyovers of all full-scale aircraft. Instead, such noise is observed only[6] in flyovers of small high-performance sailplanes having airfoils designed for large chordwise extent of laminar flow.

An alternate hypothesis[3] was based on results of tests at larger Reynolds number. In those tests, the tone was found to disappear when Reynolds number was increased such that the pressure surface boundary layer became turbulent. If Strouhal number based on laminar boundary-layer thickness were constant, tone frequency would be predicted to vary with velocity to the 1.5 power and airfoil chord to the -0.5 power.

Received August 16, 1974; revision received October 3, 1974.

Index category: Aircraft Noise, Aerodynamics (Including Sonic Boom).

*Senior Consulting Engineer, Aerodynamics. Associate Fellow AIAA.

Approximately this velocity dependence was noted experimentally[3,4] at constant chord. Also, a boundary-layer trip wire at 70 to 80% chord of the pressure surface was found in two different test programs[1,3] to eliminate tones that protruded more than 20 db above background.

It has been noted by Tam[7] that the observed[3] growth, saturation, and decay of tone strength with increasing airspeed was typical of an acoustically excited aerodynamic feedback loop. He associated this feedback with oscillations of a laminar near wake. Instead, the fluid dynamic oscillations could be Tollmein-Schlichting instability waves within the pressure-surface laminar boundary layer. As these waves are convected past the trailing edge, they would generate trailing edge noise. The resulting acoustic waves, at the frequency for which Tollmein-Schlichting waves at the trailing edge are strongest, would reinforce boundary-layer oscillations at this frequency. Then the acoustic tone frequency should be identical to the frequency of maximum-amplitude waves within the laminar boundary layer.

When amplitude of boundary-layer oscillations is measured at constant frequency and different streamwise positions, maximum amplitude is found[8] to occur at a Reynolds number given by the right branch of Shen's[9] calculated neutral stability contour. This solution was noted by Schlichting[10] to be the most rigorous solution for Tollmein-Schlichting instabilities. Assume for convenience that the airfoil pressure-surface boundary layer can be approximated by that for a flat plate. The tone frequency f should be $(2\pi)^{-1}$ times the instability angular frequency β. The frequency parameter $\beta\nu/U^2$ was given in Fig. 4 of [3] for flat plates as a function of Reynolds number based on displacement thickness. It is more instructive to express the frequency in terms of reduced frequency based on laminar displacement thickness.

$$f = (2\pi)^{-1} \; (\beta\delta^*/U) \; (U/\delta^*) \tag{1}$$

Expressing displacement thickness δ^* in terms of airfoil chord c,

$$f = (2\pi)^{-1} \, (1.73)^{-1} \, (\beta\delta^*/U) U^{3/2} \, (c\nu)^{-1/2} \tag{2}$$

The reduced frequency $\beta\delta^*/U$ calculated from[9] is given in Fig. 1. It decreases from roughly 0.17 to 0.10 as Reynolds number based on chord is increased from 1×10^5 to $2 \times 10.^6$ It is approximately constant over limited ranges of Reynolds number, yielding the form of the equation given in Ref. 3.

Measured tone frequencies for an NACA 0012 airfoil at Reynolds numbers from 0.3×10^6 to $1.4 \times 10,^6$ taken from Fig. 5 of Ref. 3, are compared in Fig. 2 with those calculated from the previous equation. Predicted frequencies are in good agreement with data. The reduced frequency is roughly equal to 0.12 at an average Reynolds number of $1 \times 10,^6$ yielding the numerical factor given in Ref. 3. Discontinuities occurred when integer multiples of wavelength were equal to the chord. Measured tone frequencies for a markedly different airfoil at Reynolds numbers form 0.1×10^6 to 0.2×10^6 were found[4] to

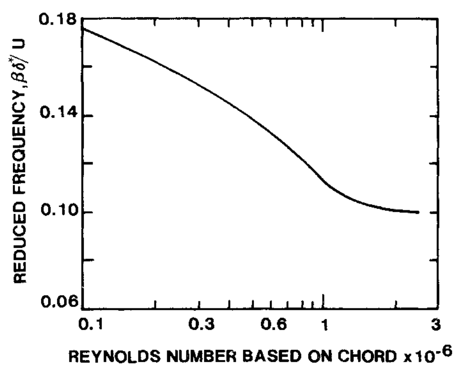

Fig. 1 Calculated reduced frequencies for neutral stability.